Submatrix Approach to Stiffness Matrix Correction Using Modal Test Data

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An efficient method for stiffness matrix correction to match modal testing data is presented. The correction is performed while preserving the connectivity of the stiffness matrix and the consistency between the stiffness matrix and the corresponding physical configuration of the structure. Significant reduction of unknown parameters is achieved by grouping the elements of the same stiffness characteristics and representing them as a multiplication of a scaling factor and a submatrix. The correction of the stiffness matrix is performed by adjusting the scaling factors. A concise formulation to identify the scaling factors is found by utilizing a least squares solution that is computationally efficient. The formulation also incorporates a capability of reducing finite element mass and stiffness matrices to test degrees-of-freedom for a direct analysis/experiment correlation. The stiffness matrix identified by this formulation produces the measured mode data accurately. The theoretical development along with a numerical example is presented to illustrate the general procedure and to demonstrate the capability of the method.

Introduction

FUTURE large space structures, such as the space stations, are envisioned to have fairly different structural dynamic characteristics compared to conventional satellite structures. The lightweight, large-scale structures are expected to be flexible, and this flexibility produces an environment for significant control/structure interactions.[1] In order to meet the stringent requirements of pointing, maneuvering, and vibration suppression under such an environment, the identification of accurate structural dynamic characteristics and dynamic models that can represent these characteristics is required.

An approach to this task is to improve the analytical dynamic model, which is often created by the finite element method, based on the experimentally identified modal characteristics. Numerous analytical model improvement techniques are available in the literature. The techniques can be classified based on the ways of defining the structural model and parameters, utilizing the experimental data, and estimating the parameters.[2] The scope of this paper is limited to the methods that use measured mode data to improve discretized analytical models. The discussion is also limited to the identification methods that utilize either submatrix scaling factors or matrix coefficients as parameters to match the experimental results. Bibliographies on various methods are available in Ref. 2.

Baruch and Bar Itzhack[3,4] developed the formulations to correct the stiffness and flexibility matrices based on the orthogonalized modes that were computed optimally from the measured modes, assuming that the mass matrix was correct. Optimal solution was obtained by using Lagrange multipliers for minimizing the changes in the matrices to satisfy specified constraints. Berman[5] showed that an analytical mass matrix can be improved by finding the smallest changes that make a set of measured modes orthogonal, using a pseudoinverse algorithm. Berman and Nagy[6] subsequently produced the analytical model improvement (AMI) procedure to correct the mass and stiffness matrices by combining the approach of Ref. 5 for the mass matrix correction and that of Ref. 3 for the stiffness matrix correction. This AMI method has a simple and noniterative computational procedure and the improved model predicts the measured mode shapes and frequencies accurately. However, the physical significance of the adjusted matrices is not preserved. The adjusted mass and stiffness matrices often do not resemble the true mass and stiffness matrices. Chen and Garba[7] also proposed a process utilizing the matrix perturbation theory to adjust the analytical model.

Recently, Kabe[8,9] presented a stiffness matrix adjustment technique using the constrained minimization theory. The adjusted stiffness matrix predicts the measured mode data accurately. Also, the connectivity of the original stiffness matrix is preserved. But the mathematical solution of the method requires rather large eigensolutions that would be prohibitively expensive for large space structure problems with thousands of degrees-of-freedom (DOF). Also, the number of unknown parameters increases as the size of the stiffness matrix increases. In order to avoid the eigensolution requirements without losing the advantage of Kabe's method, Kammer proposed the projector matrix (PM) method.[10] A more simple mathematical formulation of the method, which uses the projector matrix theory and the Moore-Penrose generalized inverse, results in a more computationally efficient solution.

White and Maytum[11] suggested a set of scaling factors that are multiplied by the prescribed submatrices to improve the accuracy of the global model. In order to identify the scaling factors, they made an approximation that uses an analytical modal matrix in place of a true (measured) modal matrix. A formulation was proposed using the concept of modal kinetic and strain energy. The energy distribution matrix, which in general contains both modal strain and kinetic energy of an initial model, allows direct examination of the modal energy among the various element groups. A review of the energy distribution provides assistance in selecting modes to be used for the scaling factor identification. The resulting mass and stiffness matrices preserve not only connectivity but also physical consistency between the matrices and the structural model. A computationally efficient pseudoinverse solution was employed to find the scaling factors. However, the adjusted matrices computed by this method do not produce the measured mode data accurately when the analytical mode shapes, which often deviate from the measured mode shapes, are used for computation. In order to achieve acceptable correlation results, the method may require an iterative process.

In this paper, a new stiffness matrix correction method is proposed in order to improve the identification accuracy of

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White/Maytum's approach while maintaining its advantages. In order to avoid using analytical mode data for correction and to provide a direct analysis/experiment correlation, an analytical model reduction procedure is incorporated as part of the correction process.

Description of the Method

Several guidelines need to be observed to successfully improve the analytical model of large structures. First, the mode data from the experimental measurements of structural behavior should be sufficient and accurate to reflect the true characteristics of the structure. Second, the number of parameters being estimated should be kept small. As the number of parameters increases, most of the estimation techniques require extensive computational capability that would be costly or even prohibitive. Reduction of the number of parameters will minimize the number of required measured modes for identification. Third, the changes of the stiffness matrix should not imply nonexistent load paths in the structure, i.e., structural connectivity should be maintained. Fourth, physically inherent internal consistency of the structural model should be preserved, i.e., the corrected stiffness matrix should be consistent with the structural model. The concept of "internal consistency" will be explained in detail in the next section. Fifth, the corrected model should be able to predict the measured mode data accurately. Finally, the mathematical algorithm should not require iterations or eigensolutions that can be computationally expensive or even prohibitive.

In order to conform to these guidelines, the method in this paper utilizes the submatrix scaling technique and the least squares approach. Since large structures can be treated as an assembly of rods, beams, plates, and shells, the structures can be efficiently parameterized by using the concept of submatrix scaling. In this technique, a parameter common to a number of elements in the structure is chosen for estimation. Changes to the parameter are made uniformly to all of the elements that have this parameter in common. For example, in the case of a planar truss, all longitudinal elements, all transverse elements, and all diagonal elements would be grouped together, thus requiring only three stiffness parameters to be identified. The reduction of parameters by grouping the elements of the structure with the same stiffness characteristics provides a significant reduction in the number of modes required for identification. Since the lower frequency structural modes are usually easy to identify accurately, computational inaccuracy due to less accurate higher modes can be minimized. This technique also preserves the consistency between the stiffness matrix and the model of the structure during the identification process. A physically realizable stiffness matrix can be obtained even when the improved stiffness matrix is not exact due to inaccurate measured mode data. Computationally, the least squares approach is used to avoid iteration and expensive eigenanalysis. Unlike the formulation proposed by White/Maytum, the procedure in this paper does not use the analytical modal matrix to compute the parameters. The procedure incorporates a static or Guyan reduction procedure in order to reduce finite element DOF to test DOF. Thus, measured mode data can be applied directly, and the corrected stiffness matrix produces the measured mode data accurately.

Theoretical Development

The equation of motion governing the response of an undamped n DOF linear system is represented by

$$(M_n)\{\ddot{u}_n\} + (K_n)\{u_n\} = \{f_n\} \tag{1}$$

where $[M_n]$ and $[K_n]$ are the system mass and stiffness matrices, respectively, $\{u_n\}$ is a vector containing physical displacements, and $\{f_n\}$ is an external excitation vector. Finite element analysis is often used to generate this discretized analytical model of the system. The corresponding eigenvalue problem becomes

$$(K_n)(\Phi_n) = (M_n)(\Phi_n)(\omega_n^2) \tag{2}$$

where (Φ_n) is the system modal matrix and will in general be a rectangular matrix of dimension $n \times l$; and (ω_n^2) is a diagonal matrix containing l system eigenvalues along its diagonal; the quantity l is the number of modes used in the above representation.

In this paper, the mass matrix is assumed to be known accurately and the stiffness matrix is supposed to be adjusted to match the measured mode data. Using the concept of submatrix scaling factors,[11] the stiffness matrix is expanded into a linear sum of submatrices, such as

$$(K_n) = (K_a) + \sum_{j=1}^{p} s_j(K_j) \tag{3}$$

where (K_a) is the initial or analytical stiffness matrix, which contains an initial estimate of the stiffness matrix, (K_j) the jth submatrix transformed into the global coordinate, s_j the jth scaling factor, and p the total number of submatrices. The submatrices can represent a single element or a group of elements of the structure having the same assumed geometry, material properties, boundary conditions, and modeling assumptions. In finite element analysis, a global stiffness matrix is assembled by a process of transforming and expanding the element stiffness matrices in local coordinates into a global coordinate system.[12] Submatrices can be easily generated during this inherent assembly process of finite element analysis. The correction of the analytical stiffness matrix is performed by adjusting the submatrix scaling factors. The use of submatrices described in Eq. (3) preserves the consistency between the stiffness matrix and the physical configuration of the structure. No unmodeled load path occurs as a result of the identification process and the corrected stiffness matrix is physically realizable. Also, a significant reduction of parameters can be achieved by judiciously grouping the structural elements with the same stiffness characteristics.

To illustrate the concept of submatrices, scaling factors, and the "consistency" of the stiffness matrix, consider the mass-spring system in Fig. 1. Assume that the mass matrix is known accurately and the stiffness matrix is to be adjusted to match measured mode data. The element stiffness matrices represented in the global coordinate system (x_1-x_2-x_3) are easily identified as

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4}$$

for the spring elements k_1, k_2, and k_3, respectively. Since the spring elements k_1 and k_3 have the same spring constants, they are grouped together to form a submatrix, (K_1), along with the

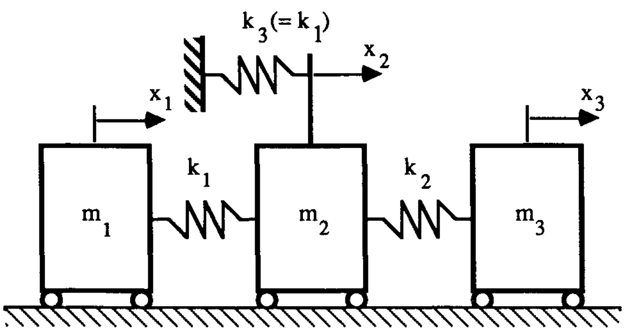


Fig. 1 A simple mass-spring system.

submatrix, (K_2), for the spring element k_2, as

$$(K_1) = k_1 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (K_2) = k_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{5}$$

A sum of the two submatrices results in the analytical stiffness matrix

$$(K_a) = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & 2k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \tag{6}$$

The stiffness matrix has five nonzero upper triangular elements that in general constitute parameters to be adjusted to match measured mode data. Instead of using each coefficient, the stiffness matrix can be adjusted by two scaling factors with known submatrices. Note that the number of parameters has been reduced from five to two. Using Kabe's assumption,[8] such that all zero coefficients of the stiffness matrix should remain zero, load paths between DOF can be preserved. However, the results based on this assumption could produce physically unrealizable stiffness coefficients. For example, the condition $(K_a)_{1,1} = -(K_a)_{1,2} = k_1$ should be preserved in order to have a physically meaningful stiffness matrix, since there is only one spring element connecting the two DOF. Also, $(K_a)_{3,3} = -(K_a)_{2,3} = k_2$ and $(K_a)_{2,2} = 2k_1 + k_2$ should be preserved. This "consistency" between the stiffness matrix and the configuration of the structure is enforced by employing submatrices. No additional constraint equations are required to impose the consistency. Further discussion will be given in the next section.

In general, the finite element model in Eq. (1) possesses a large number of DOF in comparison to the number of sensors or test DOF used in a modal survey. Thus, the finite element modes cannot be compared directly to the measured modes. In order to eliminate this problem, two different approaches have been used. The first approach is to dynamically expand each of the measured modes from test size to analysis size using the finite element mass and stiffness matrices.[13] The second approach, which is employed more frequently, is to reduce the finite element mass and stiffness matrices to the test DOF. The reduced representation, referred to as a test analysis model (TAM), provides a means to perform test/analysis correlation. Typically, the TAM is developed using a static or Guyan reduction.[14]

A stiffness matrix adjustment procedure is proposed here, based on a test analysis model. The approach of expanding the measured modes into the size of finite element DOF can be considered as the special case when the test DOF and the finite element DOF are the same. In order to obtain a TAM, the mass and stiffness matrices and the submatrices are required to be reduced to the test DOF. The modal matrix in Eq. (2) can be partitioned into two complementary sets, (Φ_{na}) and (Φ_{nd}), such as

$$(\Phi_n) = (B) \begin{Bmatrix} (\Phi_{na}) \\ (\Phi_{nd}) \end{Bmatrix} = (B)(\Phi'_n) \tag{7}$$

where (Φ_{na}) contains r active or independent DOF which is to be retained in the test analysis model and (Φ_{nd}) contains the dependent DOF to be eliminated from the representation. The (B) matrix has 1's and 0's as its coefficients and is used to define the relationship between (Φ_n) and (Φ'_n). Substitute Eqs. (7) and (3) into Eq. (2) and premultiply by $(B)^T$ to obtain

$$\left[(K_{aB}) + \sum_{j=1}^{p} s_j (K_{jB}) \right] (\Phi'_n) = (M_{nB})(\Phi'_n)(\omega_n^2) \tag{8}$$

where $(K_{aB}) = (B)^T(K_a)(B)$, $(K_{jB}) = (B)^T(K_j)(B)$, and $(M_{nB}) = (B)^T(M_n)(B)$. The transformation in Eq. (8) reorganizes the coefficients in the mass and stiffness matrices and the submatrices according to the DOF in (Φ'_n). Since the test analysis model is generated based on the initially defined analytical stiffness matrix, the transformation matrix for reduction does not incorporate the summation term in Eq. (8). Thus, the above equation is readily written in a partitioned form

$$\begin{bmatrix} (K_{aB}^{aa}) & (K_{aB}^{ad}) \\ (K_{aB}^{da}) & (K_{aB}^{dd}) \end{bmatrix} \begin{Bmatrix} (\Phi_{na}) \\ (\Phi_{nd}) \end{Bmatrix} = \begin{bmatrix} (M_{nB}^{aa}) & (M_{nB}^{ad}) \\ (M_{nB}^{da}) & (M_{nB}^{dd}) \end{bmatrix} \begin{Bmatrix} (\Phi_{na}) \\ (\Phi_{nd}) \end{Bmatrix} (\omega_n^2) \tag{9}$$

where the matrices (K_{aB}) and (M_{nB}) are partitioned according to the DOF in (Φ_{na}) and (Φ_{nd}). The Guyan reduction is employed to reduce the above equation. Define a transformation

$$(\Phi'_n) = (T)(\Phi_{na}) \tag{10}$$

where

$$(T) = \begin{bmatrix} (I_{aa}) \\ -(K_{aB}^{dd})^{-1}(K_{aB}^{da}) \end{bmatrix} \tag{11}$$

in which (I_{aa}) is an $r \times r$ identity matrix. A different reduction technique, such as the modal reduction method,[15] can be easily incorporated by using the corresponding transformation in place of the transformation for the Guyan reduction, Eq. (10). Substitution of Eq. (10) into Eq. (8) results in the reduced representation

$$(K_{nR})(\Phi_{na}) = (M_{nR})(\Phi_{na})(\omega_n^2) \tag{12}$$

where

$$(K_{nR}) = (K_{aR}) + \sum_{j=1}^{p} s_j (K_{jR}) \tag{13}$$

in the above equations, $(K_{aR}) = (T)^T(K_{aB})(T)$, $(K_{jR}) = (T)^T \times (K_{jB})(T)$, and $(M_{nR}) = (T)^T(M_{nB})(T)$. The mass, stiffness, and modal matrices and the submatrices have been reduced to test DOF $(r)(r \leq n)$. This reduced representation, or TAM, is an approximation of the finite element model in Eq. (2), since the Guyan reduction has been employed. If a measured modal matrix expanded in the finite element DOF (n) is used instead of a TAM, the steps from Eq. (7) through Eq. (13) are not required.

The measured modal matrix, (Φ_m), and measured frequencies, (ω_m^2), are now substituted into Eq. (12) and used for the stiffness matrix correction. Rearrange Eq. (12) to obtain

$$\sum_{j=1}^{p} s_j (L_j) = (R) \tag{14}$$

where

$$(L_j) = (K_{jR})(\Phi_m) \tag{15}$$

$$(R) = (M_{nR})(\Phi_m)(\omega_m^2) - (K_{aR})(\Phi_m) \tag{16}$$

The size of matrices (L_j) and (R) is $r \times m$ and the matrices are known. The quantity m is the number of modes used for the correction. The least squares method is employed to solve Eq. (14). Define a vector

$$\{s\} = \{s_1 \; s_2 \; \ldots \; s_p\}^T \tag{17}$$

whose elements are scaling factors. By equating each column of Eq. (14) and rearranging, we obtain

$$(L)\{s\} = \{r\} \tag{18}$$

where

$$(L) = \begin{bmatrix} \{L_1\}_1 & \{L_2\}_1 & \cdots & \{L_p\}_1 \\ \{L_1\}_2 & \{L_2\}_2 & \cdots & \{L_p\}_2 \\ \vdots & \vdots & & \vdots \\ \{L_1\}_m & \{L_2\}_m & \cdots & \{L_p\}_m \end{bmatrix} \quad (19)$$

$$\{r\} = \{\{R\}_1^T\{R\}_2^T \ldots \{R\}_m^T\}^T \quad (20)$$

The vector $\{L_j\}_k (j = 1,2,\ldots,p;\ k = 1,2,\ldots,m)$ represents the kth column of the submatrix (L_j) and the vector $\{R\}_k$ is the kth column of the matrix (R). Note that the sizes of (L) and $\{r\}$ are $q \times p$ and $q \times 1$, respectively, where $q = rm$. If the product of the number of test DOF and the number of measured modes is greater than or equal to the number of scaling factors to be identified, i.e.,

$$q \geq p \text{ or } m \geq p/r \quad (21)$$

the solution of Eq. (18) can be obtained by utilizing a least squares approach

$$\{s\} = [(L)^T(L)]^{-1}(L)^T\{r\} \quad (22)$$

This method is not computationally intensive because a relatively small matrix of order $p([L]^T[L])$ must be inverted. These identified scaling factors are substituted into either Eq. (13) to improve the reduced stiffness matrix or Eq. (3) to improve directly the finite element stiffness matrix. A process of expanding the corrected reduced stiffness matrix into the finite element DOF is thus avoided.

The accuracy of the adjusted stiffness matrix depends on the accuracy of the reduced representation in Eq. (12) and the measured mode data. If a sufficient number of modes can be identified with approximately the same accuracy, an inclusion of as many modes would, in general, increase the accuracy of the least squares approximation and, in turn, the corrected stiffness matrix. However, the accuracy of the stiffness matrix will at most be improved up to the accuracy of the test analysis model.

Compared to White/Maytum's method, one difference in the formulation developed above is that the measured mode data are incorporated directly without approximation. White and Maytum[11] suggested that the analytical modal matrix of the initial model can be used to approximate the measured modal matrix. In order to determine the modes to be used for the submatrix scaling factor identification, the modal strain energy distribution review was proposed. However, when the differences between the analytical and the measured mode data are not small, the solution obtained by White/Maytum's method with this approximation produces inaccurate results. Another difference is the mode data requirements for identification. Assuming that the analytical mode shapes are used to approximate the measured mode shapes, White/Maytum's method requires p (number of scaling factors) analytical modes and p measured frequencies. On the other hand, the present method requires m analytical modes and m measured frequencies, where m is the smallest integer that satisfies Eq. (21). When the number of test DOF (r) is large, the computational advantage of the present method will be significant.

Numerical Example

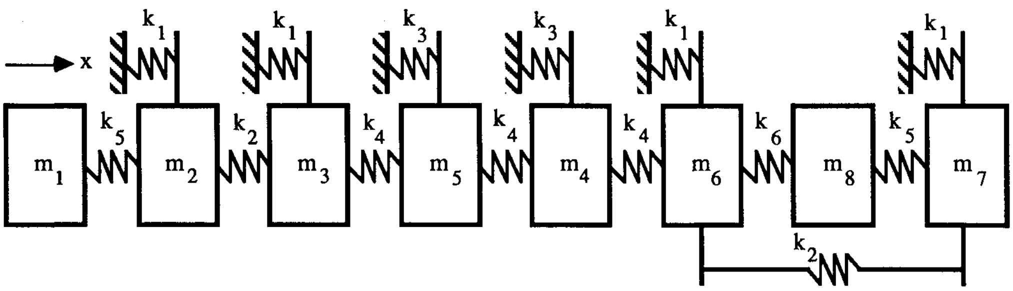
In order to illustrate the procedure and to demonstrate the capability of the stiffness matrix correction method, a numerical example used by Kabe[8] is chosen. The features of the present method will be compared with those of Kabe's and White/Maytum's procedures.

Consider an eight DOF mass-spring system with the exact values of masses and spring constants, as shown in Fig. 2. The masses are constrained to translate in the x-direction only. The dynamics of the structure is characterized by closely spaced natural frequencies, often encountered in large space structures. In this example, the mass matrix is assumed to be known exactly and the analytical stiffness matrix is to be corrected using measured mode data. The nonzero upper triangular stiffness matrix coefficients of the structure are shown in Table 1. The conditions under the heading of "consistency conditions" should be satisfied during the stiffness matrix correction procedure in order to preserve the consistency between the structural model and the stiffness matrix. These consistency conditions are automatically maintained by using the submatrix scaling factor identification approach. Kabe's method does not, in general, maintain the consistency. The spring constants are presented in Table 2. Analytical spring constants represent the corrupted values that were used to model the structure. Note the large magnitude of differences between the analytical and the exact spring constants. The stiffness matrix coefficients in Table 1 can be constructed by using the consistency conditions and the spring constants in Table 2.

To proceed with the stiffness matrix correction procedure, the submatrices of Eq. (3) should be defined. Since there are six distinct spring elements, six submatrices are defined. The nonzero upper triangular submatrices expanded to global coordinates are shown in Table 3. The normalized coefficients of each submatrix represent the coefficients divided by the spring constant of each spring. Thus, Eq. (3) can be written as

$$(K_n) = (K_a) + \sum_{j=1}^{6} s_j(K_j) \quad (23)$$

where (K_a) is composed of the analytical stiffness matrix coefficients in Table 1 and (K_j)'s are given in Table 3. Now, the problem is to identify six unknown scaling factors, s_j, in order for the corrected stiffness matrix $[K_n]$ to produce measured mode data accurately. Since Kabe's method requires identification of sixteen nonzero upper triangular stiffness coefficients, considerable reduction of parameters has been achieved. For large structures this reduction of parameters would provide a significant advantage.



$m_1 = 0.001$, $m_i = 1.0$ (i = 2 through 7), and $m_8 = 0.002$

$k_1 = 1000$, $k_2 = 10$, $k_3 = 900$, $k_4 = 100$, $k_5 = 1.5$, and $k_6 = 2.0$

Fig. 2 A one-dimensional mass-spring system.[8]

Table 1 Nonzero upper triangular stiffness matrix coefficients obtained using exact normal modes

Coefficient location in stiffness matrix	Analytical stiffness coefficients	Corrected stiffness coefficients using one, two, or three modes	Exact stiffness coefficients	Consistency conditions
1,1	2.0	1.5	1.5	k_5
1,2	− 2.0	− 1.5	− 1.5	$-k_5$
2,2	1512.0	1011.5	1011.5	$k_1 + k_2 + k_5$
2,3	− 10.0	− 10.0	− 10.0	$-k_2$
3,3	1710.0	1110.0	1110.0	$k_1 + k_2 + k_4$
3,5	− 200.0	− 100.0	− 100.0	$-k_4$
4,4	850.0	1100.0	1100.0	$k_3 + 2k_4$
4,5	− 200.0	− 100.0	− 100.0	$-k_4$
4,6	− 200.0	− 100.0	− 100.0	$-k_4$
5,5	850.0	1100.0	1100.0	$k_3 + 2k_4$
6,6	1714.0	1112.0	1112.0	$k_1 + k_2 + k_4 + k_6$
6,7	− 10.0	− 10.0	− 10.0	$-k_2$
6,8	− 4.0	− 2.0	− 2.0	$-k_6$
7,7	1512.0	1011.5	1011.5	$k_1 + k_2 + k_5$
7,8	− 2.0	− 1.5	− 1.5	$-k_5$
8,8	6.0	3.5	3.5	$k_5 + k_6$

Table 2 Spring constants obtained using exact normal modes

Spring ID	Analytical spring constants	Corrected spring constants using one, two, or three modes	Exact spring constants
k_1	1500.0	1000.0	1000.0
k_2	10.0	10.0	10.0
k_3	450.0	900.0	900.0
k_4	200.0	100.0	100.0
k_5	2.0	1.5	1.5
k_6	4.0	2.0	2.0

Table 3 Nonzero upper triangular coefficients of submatrices in global coordinates

$[K_1]$		$[K_2]$		$[K_3]$		$[K_4]$		$[K_5]$		$[K_6]$	
A[a]	B[b]	A	B	A	B	A	B	A	B	A	B
2,2	1.0	2,2	1.0	4,4	1.0	3,3	1.0	1,1	1.0	6,6	1.0
3,3	1.0	2,3	−1.0			3,5	−1.0	1,2	−1.0	6,8	−1.0
				5,5	1.0						
6,6	1.0	3,3	1.0			4,4	2.0	2,2	1.0	8,8	1.0
						4,5	−1.0				
7,7	1.0	6,6	1.0					7,7	1.0		
						4,6	−1.0				
		6,7	−1.0					7,8	−1.0		
						5,5	2.0				
		7,7	1.0					8,8	1.0		
						6,6	1.0				

[a]A: Coefficient location. [b]B: Normalized coefficients.

In order to compare directly with the results produced by Kabe,[8] assume that the test DOF of the example is the same as the finite element DOF, i.e., $r = n$. Results obtained by using a TAM will be discussed later. The number of DOF of the structure is eight, i.e., $n = 8$ and the number of parameters is six, i.e., $p = 6$. From the condition in Eq. (21) with $r = n$, we have $m \geq p/n = 0.75$. Thus, the required minimum number of measured modes for identification is one. With the number of exact normal modes greater than or equal to one, the method will produce the exact stiffness matrix.

Table 4 shows the normal mode shapes of the analytical model and the exact model. Since these mode shapes are very different, White/Maytum's approximation, which uses the analytical mode shapes instead of the measured mode shapes, would produce inaccurate results. In order to determine the modes to be used for the submatrix scaling factor identification using White/Maytum's method, the modal strain energy distribution of each submatrix is examined. Table 5 shows the fractional modal strain energy distribution. The entries of the table are obtained by evaluating

$$k_{a_j}\{\Phi_a\}_i^T(K_j)\{\Phi_a\}_i \times 100/\omega_{a_i}^2$$

$$\text{for } i = 1,2,\ldots,8 \text{ and } j = 1,2,\ldots,6 \qquad (24)$$

where k_{a_j} is the jth analytical spring constant, the vector $\{\Phi_a\}_i$ is the ith mass normalized analytical mode shape, and ω_{a_i} is the ith analytical frequency. The subscripts i and j correspond to the mode and submatrix number, respectively. Since submatrix 2 has negligible strain energy for all the modes, its scaling factor is eliminated from identification to best fit the measured frequencies. Thus, for this example, White/Maytum's method requires five modes for identification. By examining the largest contribution of the modal strain energy for each mode and submatrix, five modes (1, 2, 4, 7, and 8) are selected for identification. If the present method is used and the measured mode shapes are approximated by the analytical mode shapes, the minimum data requirements are one analytical mode and one measured frequency. The advantage of the present method in terms of data requirements is clear. Table 6 shows the identified submatrix scaling factors with the first one, two, or three exact normal modes obtained by the present method in comparison to the scaling factors obtained by five and eight analytical mode shapes by White/Maytum's method. The exact scaling factors are identified by the present method. On the other hand, the scaling factors identified by White/Maytum's method are inaccurate due to the approximation made on the mode shapes. By including eight modes the accuracy of the scaling factor s_4 is improved. However, the accuracy of the scaling factors s_1 and s_3 has been deteriorated, suggesting that inclusion of more number of inaccurate modes does not necessarily improve the accuracy of the identification.

The corrected spring constants computed from the scaling factors obtained by the present method are shown in Table 2. Table 1 shows the corrected stiffness matrix coefficients derived from these scaling factors and Eq. (23) for the cases of one, two, or three modes. The stiffness matrix coefficients are exact as expected. The fact that the procedure requires only one mode to produce the exact stiffness matrix gives an advantage over Kabe's method, which requires three modes to obtain the exact stiffness matrix. All eight natural frequencies and mode shapes produced by the identified stiffness matrix are exact. The method in this paper is not computationally intensive because no eigensolution is necessary.

In practice, because of the limitations in the experimental apparatus and the modal parameter identification techniques, the measured modes will not be exact. However, the identification procedure should produce reliable results even when imperfect mode data are used. To study the procedure's sensitivity to imperfect data and to demonstrate the advantage of the procedure, the simulated test modes produced by Kabe[8] are used. The simulated modes were obtained so that the off-diagonal terms of the generalized mass matrix would not exceed 0.1 when the diagonal values were normalized to unity. There are two ways of using these imperfect mode data. One approach is to use these modes without any modification. The other approach is to mathematically orthogonalize the measured modes with respect to the mass matrix and then perform the identification. In this paper, both methods will be used to adjust the stiffness matrix.

The first approach is to use the simulated mode shapes directly. The simulated mode shapes used in this example are given in Table 4. Table 7 includes the corrected nonzero upper triangular stiffness matrix coefficients, using these imperfect simulated modes directly. Tables 8 and 9 show the corrected spring constants and the identified submatrix scaling factors, respectively. Because of the "corruption" in the simulated modes, one of the corrected spring constants, k_6, in Table 8 becomes physically unrealizable when one mode is used for identification. Nevertheless, the consistency of the stiffness matrix is preserved. More accurate mode data or a greater number of modes should be used to eliminate the physically unrealizable spring constant. For example, inclusion of the second mode removes this physically unrealizable value.

To use another approach, which is based on mathematical orthogonalization of mode shapes with respect to the mass matrix, the symmetric matrix orthogonalization procedure[16] is used. The orthogonalized mode shapes are defined by

$$[\Phi_c] = [\Phi_m]([\Phi_m]^T[M][\Phi_m])^{-\frac{1}{2}} \tag{25}$$

where $[\Phi_m]$ is the measured modal matrix and $[\Phi_c]$ the orthogonalized modal matrix. The orthogonalized mode shapes that were produced by Kabe[8] are given in Table 4 in comparison with the simulated modes and the exact normal modes. The

Table 4 Comparison of mode shapes

Mode number	Normal mode shapes: Analytical model	Normal mode shapes: Exact model	Simulated mode shapes[a]	Orthogonalized mode shapes[a]
1	0.002	0.144	0.029	0.112
	0.001	0.054	0.015	0.042
	0.127	0.360	0.374	0.356
	0.696	0.606	0.595	0.609
	0.696	0.605	0.604	0.600
	0.127	0.362	0.356	0.365
	0.002	0.062	0.117	0.076
	0.107	0.505	0.551	0.522
2	0.007	2.776	2.753	2.774
	0.004	0.915	0.909	0.917
	0.190	0.127	0.098	0.138
	−0.681	−0.088	−0.080	−0.078
	0.681	0.041	0.027	0.050
	−0.190	−0.124	−0.127	−0.123
	−0.004	−0.337	−0.365	−0.330
	−0.192	−0.506	−0.539	−0.498

[a]Adopted from Ref. 8.

Table 5 Fractional modal strain energy distribution in submatrices, %

Mode number	$[K_1]$	$[K_2]$	$[K_3]$	$[K_4]$	$[K_5]$	$[K_6]$
1	7.87	0.05	70.98	21.09	0.00	0.00
2	10.93	0.07	41.98	47.01	0.01	0.00
3	98.16	0.61	0.00	0.03	1.20	0.00
4	99.20	0.60	0.01	0.06	0.01	0.12
5	82.93	0.61	0.83	15.44	0.16	0.04
6	78.67	0.57	1.84	18.77	0.13	0.03
7	1.20	0.01	0.00	0.00	98.79	0.00
8	0.29	0.00	0.00	0.04	33.14	66.53

Table 6 Identified submatrix scaling factors

Scaling factor ID	Present method using one, two, or three exact normal modes	White/Maytum's method with analytical modes: Five modes	White/Maytum's method with analytical modes: Eight modes	Exact solution
s_1	−500.0	−461.8	−453.7	−500.0
s_2	0.0	0.0	0.0	0.0
s_3	450.0	463.1	372.3	450.0
s_4	−100.0	−164.6	−104.8	−100.0
s_5	−0.5	−0.5	−0.5	−0.5
s_6	−2.0	−2.0	−2.0	−2.0

Table 7 Comparison of corrected nonzero upper triangular stiffness matrix coefficients

Coefficient location in stiffness matrix	Analytical stiffness coefficients	With simulated modes, Present method: One mode	With simulated modes, Present method: Two modes	With orthogonalized modes, Present method: One mode	With orthogonalized modes, Present method: Two modes	With orthogonalized modes, Kabe's method [Ref. 8]: One mode	With orthogonalized modes, Kabe's method [Ref. 8]: Two modes	Exact stiffness coefficients
1,1	2.0	3.5	1.7	1.6	1.5	1.7	1.5	1.5
1,2	−2.0	−3.5	−1.7	−1.6	−1.5	−2.1	−1.4	−1.5
2,2	1512.0	957.9	1011.1	955.0	1011.7	1030.4	1010.9	1011.5
2,3	−10.0	−0.4	−9.3	−1.0	−10.4	−10.1	−8.0	−10.0
3,3	1710.0	977.1	1113.6	972.1	1109.8	1276.6	1091.0	1110.0
3,5	−200.0	−22.7	−104.2	−18.6	−99.7	−198.6	−88.8	−100.0
4,4	850.0	976.9	1108.2	970.1	1099.5	1235.2	1098.1	1100.0
4,5	−200.0	−22.7	−104.2	−18.6	−99.7	−178.6	−99.6	−100.0
4,6	−200.0	−22.7	−104.2	−18.6	−99.7	−198.5	−99.6	−100.0
5,5	850.0	976.9	1108.2	970.1	1099.5	1239.3	1094.0	1100.0
6,6	1714.0	974.7	1115.9	973.5	1112.8	1279.8	1113.5	1112.0
6,7	−10.0	−0.4	−9.3	−1.0	−10.4	−10.0	−11.9	−10.0
6,8	−4.0	2.5	−2.2	−1.4	−3.0	−4.1	−3.1	−2.0
7,7	1512.0	957.9	1011.1	955.0	1011.7	1002.1	1013.6	1011.5
7,8	−2.0	−3.5	−1.7	−1.6	−1.5	−2.0	−2.4	−1.5
8,8	6.0	1.0	3.9	3.0	4.5	5.1	4.4	3.5

Table 8 Comparison of corrected spring constants

Spring ID	Analytical spring constants	With simulated modes: Present method, One mode	With simulated modes: Present method, Two modes	With orthogonalized modes: Present method, One mode	With orthogonalized modes: Present method, Two modes	With orthogonalized modes: Kabe's method, One or two modes	Exact spring constants
k_1	1500.0	954.0	1000.1	952.4	999.7		1000.0
k_2	10.0	0.4	9.3	1.0	10.4		10.0
k_3	450.0	931.5	899.8	932.8	900.2	Not	900.0
k_4	200.0	22.7	104.2	18.6	99.7	uniquely	100.0
k_5	2.0	3.5	1.7	1.6	1.5	determined	1.5
k_6	4.0	−2.5	2.2	1.4	3.0		2.0

Table 9 Identified submatrix scaling factors by the present method

Scaling factor ID	Identified by the simulated modes: One mode	Identified by the simulated modes: Two modes	Identified by the orthogonalized modes: One mode	Identified by the orthogonalized modes: Two modes	Exact solution
s_1	−546.0	−499.9	−547.6	−500.3	−500.0
s_2	−9.6	−0.7	−9.0	0.4	0.0
s_3	481.5	449.8	482.8	450.2	450.0
s_4	−177.3	−95.8	−181.4	−100.3	−100.0
s_5	1.5	−0.3	−0.4	−0.5	−0.5
s_6	−6.5	−1.77	−2.6	−1.0	−2.0

Table 10 Corrected nonzero upper triangular stifness matrix coefficients in the finite element DOF

Coefficient location in stiffness matrix	Analytical stiffness coefficients	Corrected stiffness coefficients using: Modes 1 and 2	Corrected stiffness coefficients using: Modes 2 and 3	Corrected stiffness coefficients using: Modes 1, 2, and 3	Exact stiffness coefficients
1,1	45.0	28.3	27.8	28.4	30.0
1,2	−45.0	−28.3	−27.8	−28.4	−30.0
2,2	135.0	−113.6	114.3	114.9	120.0
2,3	−50.0	−35.5	−37.1	−36.9	−40.0
3,3	110.0	105.5	107.1	107.0	110.0
3,5	−20.0	−20.2	−20.5	−20.5	−20.0
4,4	80.0	99.0	100.3	100.3	100.0
4,5	−20.0	−20.2	−20.5	−20.5	−20.0
4,6	−20.0	−20.2	−20.5	−20.5	−20.0
5,5	80.0	99.0	100.3	100.3	100.0
6,6	125.0	116.8	118.3	118.0	120.0
6,7	−50.0	−35.5	−37.1	−36.9	−40.0
6,8	−15.0	−11.3	−11.2	−11.0	−10.0
7,7	135.0	113.6	114.3	114.9	120.0
7,8	−45.0	−28.3	−27.8	−28.4	−30.0
8,8	60.0	39.5	39.0	39.5	40.0

adjusted nonzero upper triangular stiffness matrix coefficients obtained using the orthogonalized mode shapes are presented in Table 7 in comparison to those obtained by Kabe's method. The stiffness coefficients produced by the present method are closer to the exact stiffness coefficients. The adjusted stiffness coefficients using two modes are very close to the exact values. Also, Table 8 shows that the spring constants are getting closer to the exact values as the number of modes used in the identification increases. The identified submatrix scaling factors by the orthogonalized mode shapes are included in Table 9. The present method not only preserves the connectivity of the stiffness matrix but also produces consistent stiffness matrix that are physically realizable. It can be easily noticed from the results in Table 7 and the consistency conditions in Table 1 that the stiffness coefficients identified by Kabe's method are not consistent. The spring constants in Table 8 cannot be determined uniquely by using Kabe's method.

In an actual modal survey, the number of test DOF is typically less than the number of finite element DOF. In this case, the reduction of the finite element DOF to the test DOF is required for an efficient analysis/experiment correlation. In order to illustrate the stiffness matrix correction procedure using a reduced representation, Kabe's example in Fig. 2 is utilized with different mass and stiffness properties. The new mass properties are $m_1 = m_3 = m_4 = 0.05$, $m_2 = m_5 = m_6 = m_7 = 1.0$, and $m_8 = 0.5$. The exact and analytical stiffness coefficients for this example are contained in Table 10. The analytical stiffness coefficients were obtained by changing the exact stiffness coefficients up to 50%. Although the number of test DOF is often much smaller than the number of finite element DOF, in order to preserve the dynamic characteristics of the finite element model with good accuracy, five DOF (x_2, $x_5, \ldots, x_8$) are selected as the test DOF for this example. The reduced representation of Eq. (12) is readily obtained using

Table 11 Comparison of natural frequencies, rad/s

Exact stiffness matrix		Corrupted stiffness matrix		Test analysis model adjusted using			Finite element model adjusted using		
Before reduction	After reduction	Before reduction	After reduction	Modes 1 and 2	Modes 2 and 3	Modes 1, 2, and 3	Modes 1 and 2	Modes 2 and 3	Modes 1, 2, and 3
6.130	6.130	5.827	5.829	6.129	6.111	6.119	6.123	6.108	6.115
8.288	8.312	7.571	7.593	8.288	8.288	8.291	8.192	8.201	8.204
9.688	9.693	8.770	8.781	9.628	9.688	9.688	9.601	9.662	9.662
10.678	10.678	11.856	11.860	10.682	10.654	10.674	10.679	10.651	10.671
12.831	12.831	14.148	14.150	12.414	12.490	12.516	12.411	12.487	12.513
25.174	—	30.782	—	—	—	—	24.437	24.219	24.505
44.815	—	40.132	—	—	—	—	44.595	44.879	44.883
47.108	—	47.202	—	—	—	—	46.124	46.464	46.445

the Guyan reduction procedure described previously. The influence of the reduction on the natural frequencies is shown in Table 11. The frequencies of the reduced model represent the first five frequencies of the finite element model accurately for both the exact and corrupted stiffness matrices. The same submatrices given in Table 3 are used, since the same structure is examined. The number of DOF of the TAM is five, i.e., $r = 5$, and the number of scaling factors is still 6, i.e., $p = 6$. Thus, from Eq. (21), the required minimum number of measured modes becomes two. The exact normal modes are reduced to the test DOF for the stiffness matrix correction. Two (modes 1 and 2 or modes 2 and 3) and three modes (modes 1, 2, and 3) are employed for the correction. The identified scaling factors can be used to correct the stiffness matrices both in the test DOF using Eq. (13) and in the finite element DOF using Eq. (3).

The frequencies of the adjusted TAM and the finite element model are shown in Table 11, and the corrected stiffness matrix coefficients in the finite element DOF are shown in Table 10. The identified stiffness matrices still preserve the connectivity and the consistency. The corrected results using the three modes does not show appreciable improvement, indicating that the characteristics of the TAM may have strong influence on the accuracy of the adjusted stiffness matrix. The stiffness matrix coefficients are not exactly corrected because the measured modes in the test DOF are used for the correction and the reduced model is not exact. The accuracy of correction is a function of not only the accuracy of the reduced representation but also the number and location of the test DOF. Since the identification of the scaling factors is performed based on the reduced model, the error produced by the reduction procedure will not be corrected by the stiffness matrix adjustment procedure.

Concluding Remarks

An efficient procedure to update the analytical stiffness matrix using measured mode data was presented. By incorporating the concept of submatrix scaling factors and the least squares method, it was shown that the scaling factors can be identified when the product of the number of test DOF and the number of measured modes is greater than or equal to the number of the scaling factors. The resulting stiffness matrix retains physically meaningful consistency between the model of the structure and the stiffness matrix. When the measured modes are accurate, the adjusted stiffness matrix is able to produce the measured mode data accurately. The method also provides reliable results even when the measured modes fail to meet the orthogonality check, as long as the off-diagonal terms of the generalized mass matrix remain small compared to the diagonal terms. Since a reduced representation of a model or a TAM is frequently required for a direct analysis/experiment correlation, the Guyan reduction procedure has been incorporated. The accuracy of the identification using a TAM depends on the accuracy of the TAM. Thus, the generation of a reduced representation should be considered carefully in order to avoid introducing significant error during the reduction process. Although computational savings and the reduction of measurement requirements may vary, depending on the problem, significant computational efficiency is anticipated when this method is used to correct the analytical model of large structures with many stiffness elements of identical characteristics.

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